ROLE OF FRACTALS IN SCALING SPATIOTEMPORAL DATA

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ABSTRACT

Fractal theory as a means of achieving relationships between different scales, has gained popularity in last two decades. It has been proposed that the physics of atmospheric turbulence results in multifractal properties, which reflects in atmospheric phenomena, like cloud formation and rainfall. Especially, the modeling of the spatial and temporal variability of rainfall, which is very much discontinuous and violent in usual spatial-temporal scales, can gain substantially from fractal theories. In this paper, some results of multifractal analysis and modeling of spatial rainfall variability in Japan are presented. Two new modifications to the existing methods, that are intended to make the distributions more acceptable for operational purposes, are presented with some modeling results.

1. INTRODUCTION

Mostly due to the concerns of the local effects of global weather change, the interrelationships between global circulations related phenomena and the surface hydrological processes at watershed and smaller scales have become an important consideration of today. The general circulation scale analysis and forecasting tools have improved tremendously both due to the increasing sociopolitical importance of understanding the global warming and the exponentially increasing computing capabilities. Investigations on the consequences of various climatic scenarios that are predicted by global and regional climatic models, at the watershed scale require means of relating the magnitude of various processes at different spatiotemporal scales. Ideally the best means of achieving this relationship is perhaps the use of one of more of physically based atmospheric models running at increasingly smaller spatial and temporal scales, with the boundary conditions obtained from the larger scale analyses. However, in practice the operational hydrologist need more accessible and computationally economical means of downscaling of meteorological data. In response for this requirement, a number of researchers have developed stochastic means of downscaling the large-scale analyses and observations. Using fractal theory is one of such stochastic methods that gained popularity past two decades.

While the fractal theory has been used in downscaling diverse types of geophysical and meteorological measures, including earth's topography, atmospheric temperature, etc., perhaps the most popular candidate has been the modeling of rainfall and related phenomena like cloud distributions. Rainfall process is among the geophysical fields that show the highest variability and discontinuity and this fact makes it difficult to use traditional techniques of scaling like Thiessen polygons or fitting of polynomial functions, which are based on a continuous and smooth spatial or temporal variations. On the other hand, fractal-scaling theories are based on discontinuous mathematics so that they are inherently able to deal with the above complexities.

Fractal behavior that is characterized by single parameter driven relationships between scales is termed as '*simple scaling*'. Fields which needs infinite number of parameters to describe this 'scaling', due to the non-linear behavior of statistical moments, are known as '*multifractals*' and hence their scaling as '*multiple scaling*'. It is well known that the rainfall behavior is best explained by multiple scaling as opposed to simple scaling [Schertzer and Lovejoy (1987), Gupta and Waymire (1990)]. Therefore, the rest of this manuscript is devoted to rainfall modeling using multifractal theory. Mathematically, random cascade processes produce multifractal fields. (An example in 2-dimension is given in figure 1.) Richardson (1922) laid down the basic idea of scaling and conservation of energy-flux between different cascade levels. The same conservation principal is applied to model rainfall quantity as a cascade process.

In the rest of this paper, one multifractal model, directly based on random cascade model, is presented. The model is applied to spatial rainfall of Japanese archipelago measured by gauge networks and meteorological radar to validate the rainfall-scaling phenomenon. Finally, two issues that are important for using multifractals in operational purposes, namely 1) incorporating spatial heterogeneity as indicated by long-term averages of spatial rainfall and 2) preserving pixel-level temporal-correlations in fractal based spatial distributions are discussed.

2. MULTIFRACTAL MODEL

A random cascade process (figure 1) at the limit of large step sizes shows the following scaling property:

$$M(\lambda, q) = \sum_i [R_{i,\lambda}]^q = \lambda^{\tau(q)} \qquad (1)$$

where $R_{i,\lambda}$ is the value of the field at the *i*th box at the scale λ ($\lambda = L/l$ where l is scale and L is the largest scale of interest.) and $M(\lambda,q)$ is the statistical moment of order q at scale l. Thus, the power-law behavior of statistical moment with scale for given value of q, indicates fractal scaling. The curvature of the scaling exponent indicates the degree of multiple scaling (as opposed to single scaling). Over and Gupta (1996) proposed the β-*lognormal* model to represent spatial rainfall as a cascade process. The cascade generator of the model is specified as:

$$P(W = b^{\beta - \sigma^2 \frac{\log[b]}{2} + \sigma X}) = b^{-\beta} \quad ; \quad P(W = 0) = 1 - b^{-\beta} \qquad (2)$$

X is a standard normal variable; β and σ^2 are model parameters. This model treats zero-nonzero partitioning explicitly by the parameter β. The parameters of the model are estimated using the first and second derivatives of the scaling exponent $\tau(q)$ in equation 1 (Over and Gupta, 1996).

3. ANALYSIS

The Japan Meteorological Agency's Radar-AMeDAS rainfall data is an hourly spatial rainfall database measured mainly by weather radar, but later calibrated using the countrywide rain gauge network. We demonstrate a means of distributing spatial rainfall amount given as a large-scale forcing in a high-resolution spatial domain, using this rainfall product averaged to a daily scale. An area of 128x128 pixels (approximately 5kmx5km each) bounded by 39.65N 134.5W, 33.3S and 142.4375E was selected for this analysis. Firstly, the ability to represent the spatial variability using multifractal theory was tested by applying the equation 1. The statistical moment $M(\lambda,q)$ was computed for daily rainfall maps at different spatial scale, $\tilde{\lambda}$. These results were tested for power-law behavior of statistical moments (e.g. figure 2). The linearity of the relationships in log-log scale shows that the rainfall fields indeed show fractal scaling properties. Further the non-linearity of the scaling exponents $\tau(q)$ (figure 3.) indicates that the type of scaling is multiple (as opposed to simple).

4. DOWNSCALING PROCEDURE

The typical application of multifractal theory to downscale a given time-series of large-scale forcings is as follows: By means of data analysis, (as explained in the section before) some statistical relationships between large-scale forcing and values of model parameters (β and σ^2) are established. For the present case, it was found that only β shows a significant sensitivity to the large-scale forcing and hence σ^2 was considered constant. In the operational phase, given a value of large-scale forcing, it is possible to obtain suitable model parameters by following the above relationships. Then a cascade simulation of seven steps ($2^7=128$) is done using the β-*lognormal* model to draw required cascade weights.

The best approach to validate the above procedure is to use a series of observed rainfall snapshots to obtain spatially average rainfall time series and then use that series to run the model. Then the properties of the original (observed) spatial data and the spatial distributions obtained from the model can be compared. It should be noted that the above methodology is a stochastic approach and thus there is no one-to-one correspondence between the two quantities. Only ensemble statistical properties can be compared. Figure 4 shows the comparison of zero-fractions in observed and model generated distributions. The reason for the good agreement of the zero fractions is the explicit treatment of zero rainfalls in the model.

The cascade approach explained above can represent the spatial variability of rainfall as observed in real spatial rainfall snapshots and therefore produces realistic zero-fraction distributions and spatial correlations. However, this procedure does not take in to account two important spatiotemporal aspects of observed rainfall. The first is the negligence of spatial heterogeneity of rainfall as seen by long term averaged spatial rainfall observations: For example the summer rainfall in Japan (Figure 5) shows higher rainfall amount in the *kii* peninsula area compared with the north-eastern part. However, multifractal downscaling cannot treat this type of heterogeneity because the random cascade process is a homogeneous in the ensemble sense. The second is related to the negligence of the temporal correlations among the rainfall amount at a given pixel at adjacent time steps. The remainder of this paper will briefly discuss two new approaches proposed to rectify those two shortcomings.

5. SPATIAL HETEROGENEITY

We propose the following scheme to incorporate spatial heterogeneity in to cascade disaggregation. Rainfall is considered a combined effect of two processes, namely, 1) a multifractal (stochastic) process which is highly variable in space but statistically uniform over the area concerned, at least at regional and smaller scales and 2) a deterministic process that represents the heterogeneity of rainfall in space that is used to 'modify' the above multifractal process. We assume that the latter process can adequately be represented by the long-term averaged rainfall.

The following equation expresses the proposed model in mathematical notation

$$R_{i,j} = M_{i,j} G_{i,j}, \quad M_{i,j} = \begin{cases} 0 & for\ G_{i,j} = 0 \\ R_{i,j} / G_{i,j} & otherwise \end{cases} \tag{1}$$

where $R_{i,j}$ is the rainfall on the pixel (i,j) and $G_{i,j}$ is the component of that rainfall that is invariant over a long accumulation. Then, by definition, $M_{i,j}$ is a component that is randomly distributed in the space so that M yields a uniform field at large accumulations. Instead of R, M is represented by multifractal model.

An example for the scaling of field M is given on the right side of figure 2. It is obvious that field M also shows scaling properties like original spatial rainfall. However, the scaling properties of M are distinctly different from those of R. The multifractal properties and their regression relationships with large-scale forcing are now based on field M instead of R. In simulations, first a field M is constructed with cascading and then modified with long-term average field G to obtain R. From the results shown in figure 6 and 7 it is clear that the new model can indeed capture the spatial heterogeneity in addition to the spatial variability that is represented by a traditional multifractal model.

6. TEMPORAL CORRELATIONS

In order to understand the nature of temporal correlations existing among rainfall amounts of same grid-box at adjacent time steps, a correlation analysis was performed. With the basic premise that the persistence of rainfall is correlated (positively) to the cascade-level and (negatively) to the time separation between two spatial datasets, the following analysis was performed: Observed rainfall was aggregated by averaging four-pixels at a time to obtain the opposite process of that shown in figure 1. This process allows the determination of cascade weights W, for each cascade step. Then at each cascade level, a correlation analysis was performed, making the simplifying assumption that all weights at the same cascade level at a give time, correlates similarly to the corresponding weights at an adjacent time step. This results in a single autocorrelation curve for the weights at each cascade level. Figure 7 shows such curves for a selected rainstorm, observed at hourly time step.

It is clear that a Markov process of order one can sufficiently describe the correlations among cascade weights. However, since a cascade generator is required to be positive-definite, it is difficult to directly utilize this relationship to implement a correlated-cascade scheme. Instead, logarithmically transformed weights are considered. As shown in Figure 8,

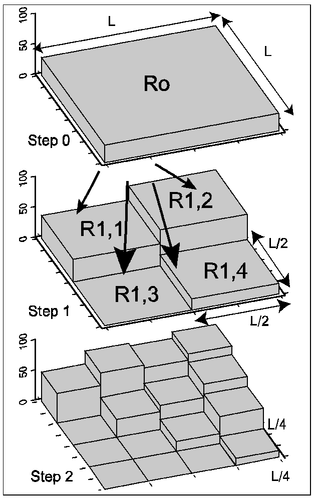

Figure 1. A random cascade process in 2-dim. At each step, a segment is divided into b ($=4$) equal parts and each part is multiplied by a value (cascade weight) drawn from a specified distribution (*generator* of the cascade).

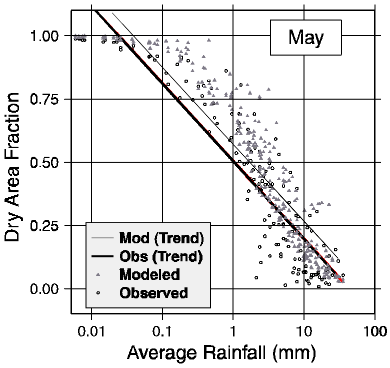

Figure 4. Dry fraction distribution of observed and simulated rainfall.

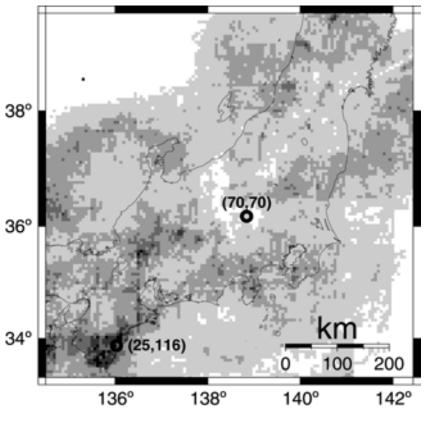

Figure 6. Quantile-quantile plots for intensity distributions of observed and modeled rainfall at two selected locations shown in figure 5.

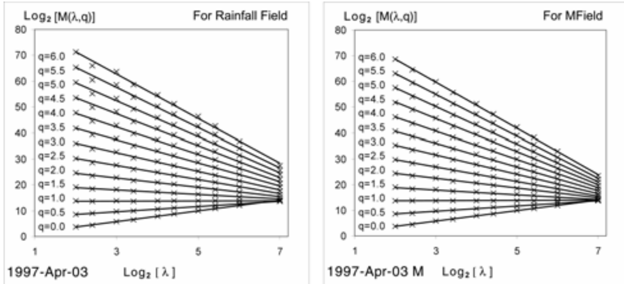

Figure 2. The scaling properties of statistical moment, $M(\lambda, q)$ for rainfall snapshots. Left: Rainfall fields. Right: Modified M fields.

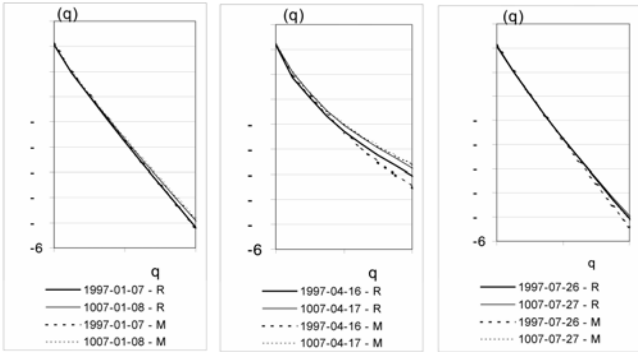

Figure 3. Estimations of $\tau(q)$. Solid lines: rainfall fields (R). Dashed lines: modified fields (M).

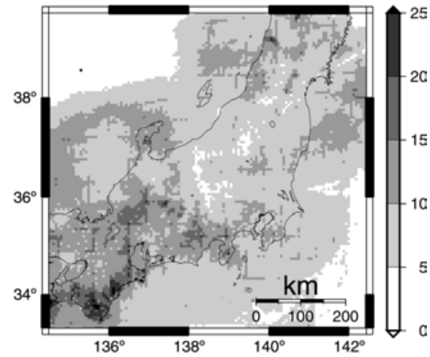

Figure 5. . Spatial heterogeneity shown by the daily average rainfall intensities for May, based on radar-AMeDAS data from 1995 to 1999. Units: mmday^{-1}

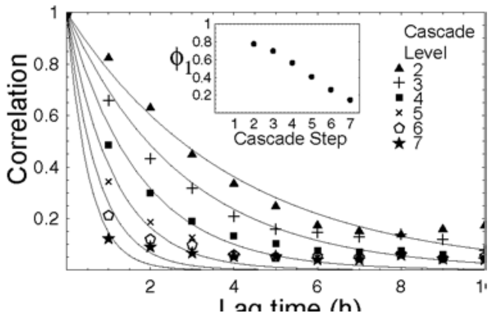

Figure 7. Correlation among cascade weights at adjacent time steps. Weights corresponding to each cascade step are analyzed separately. Inset: Variation of Markov parameter ϕ_1 ($X_i = X_{(i-1)}\,\phi_1 + Z$)

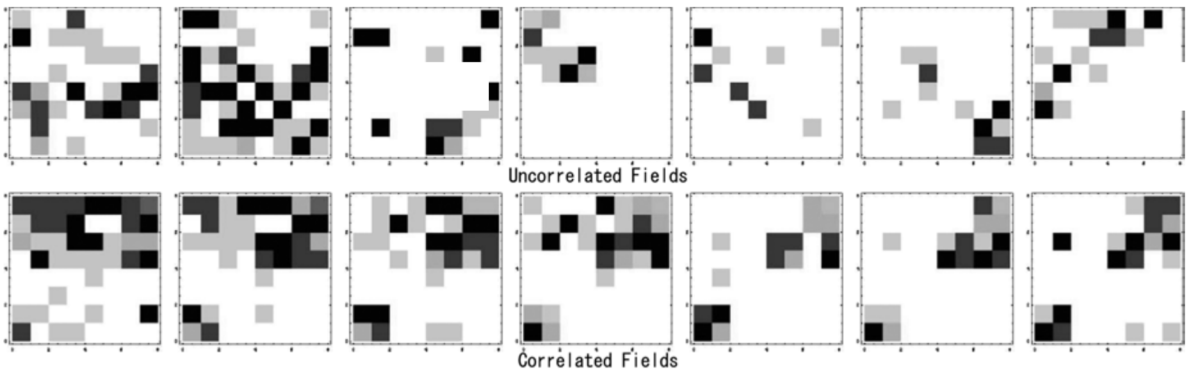

Figure 8. A selected portion of a time-series of cascade simulated rain field. Top: Typical cascade disaggregation. Bottom: Cascade with proposed correlation scheme. Note the persistence of rainfall in the bottom series.

these transformed weights also show excellent correlation structure, which can be modeled as a Markov Process. Utilizing the lognormal correlations, the following scheme can be proposed to analyze and model a time-series of spatial rainfall snapshots. The zero weighs can also be correlated by considering the conditional probabilities of occurrence of non-zero weight, with the occurrence of zero or non-zero weight at the previous time step. The mathematical formulation and most of the results are skipped in the present manuscript due to the space limitation. Figure 9 shows a selected sample of rainfall snapshots simulated using the proposed correlation scheme together with a sample from a cascade generation without considering the correlations. It clearly shows that the proposed correlation scheme improves the spatiotemporal behavior of simulated rainfall by mimicking the persistence found in observed storm data.

7. DISCUSSION

The main strength of cascade schemes in the context of spatial rainfall disaggregation is their inherent ability to represent high degree of variability and discontinuity. However, the products of using pure cascade process in two (spatial) dimension lack two important properties that are evident in observed spatial rainfall. They are 1) spatial heterogeneity revealed from long-term averaged spatial fields and 2) persistence of rainfall between adjacent time scales. In this paper we describe means of modifying cascade schemes to achieve latter two properties while retaining the former. Without directly dealing with dynamics of rainfall generation, these stochastic methods provide a means of mimicking the properties of rainfall of high spatiotemporal resolution.

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